Rational endomorphisms on Fano hypersurfaces

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joint work with David Stapleton

◇ Zoom Birational Geometry Seminar ◇

Maps

Given an arbitrary smooth projective variety X, what sort of morphisms can one write down?

$$\mathrm{id} : X \to X$$

$$X \to \mathrm{pt}.$$

Ex. $X \subset \mathbb{P}^N \implies \exists X \to \mathbb{P}^n$ of every dimension $n \geq \dim X$.

Q: What about other *rational* maps?

Rational self-maps

Suppose X is *unirational* of dimension n:

$$\exists \text{ dominant map } \mathbb{P}^n \dashrightarrow X.$$

Then we can precompose to get a rational self-map

$$X \dashrightarrow \mathbb{P}^n \dashrightarrow X.$$

This gives many elements in the *set*

$$\begin{array}{c} \mathrm{RatEnd}(X) = \{\text{dominant } X \dashrightarrow X\} \supset \mathrm{Bir}(X) \\ \downarrow \scriptstyle\deg \\ \mathbb{Z} \end{array}$$

Longstanding open problem: does there exist a smooth rationally connected variety which is not unirational?

Observation: if X does NOT admit any $\phi \in \mathsf{RatEnd}(X)$ with $\deg \phi \geq 2$, then X is not unirational!

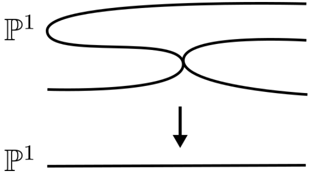
Q. What sort of varieties admit rational endomorphisms?

Q. Obstructions to the existence of

$$\varphi \in \mathsf{RatEnd}(X)$$

with $\deg(\phi) \geq 2$?

Examples



Ex. $\mathbb{P}^1$ has lots of rational endomorphisms of all possible degrees.

Ex. Let E = elliptic curve.

$\mathrm{End}(E) = \mathbb{Z}$: up to translation have

$$\varphi_k \colon E \xrightarrow{\cdot k} E \implies \deg \phi_k = k^2$$

$\mathrm{End}(E) = \mathbb{Z}[i]$. Then $a + bi \in \mathbb{Z}[i]$ corresponds to

$$\mathrm{End}(E) \ni \phi_{(a+bi)} \colon E \to E \text{ of degree } a^2 + b^2.$$

Fact: If C is a curve of genus $g \geq 2$, then $\mathrm{RatEnd}(C) = \mathrm{Aut}(C)$.

Higher dimensions

Rational (or even ruled) varieties admit rational endomorphisms of every degree.

Abelian surfaces: depends on $\mathrm{End}(A)$.

Q. Let S be a K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}$. Does S admit rational self-maps of degree ≥ 2?

- Partial work by Dedieu for K3s: any such map must have degree equal to a perfect square k^2.

Theorem (Kobayashi-Ochiai)

If X is a general type variety, then $\mathrm{Bir}(X) = \mathrm{RatEnd}(X)$.

Main result

$$X_d \subset \mathbb{P}^{n+1}_{\mathbb{C}}$$

a (very general) smooth hypersurface of dimension n and degree d.

Focus on degree $d \leq n+2$ (Fano and Calabi-Yau range).

Today: Finding obstructions to the existence of rational maps of certain degrees using specialization to char p techniques.

Theorem (C-Stapleton)

$X_d \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ very general, $n \geq 3$. Let p be a prime number such that

$$d \geq p \left\lceil \frac{n+3}{p+1} \right\rceil .$$

If $\phi \in \operatorname{RatEnd}(X)$ has degree λ, then $\lambda \equiv 0$ or $1 \pmod{p}$.

Graph

Beheshti-Riedl: For a fixed degree d, hypersurfaces X_d of large enough dimension are unirational. So these will have lots of rational endomorphisms.

Analogy:

Fano hypersurfaces of large degree behave very differently from those of small degree.

Rationality of Fano hypersurfaces

Starting point: $X_d \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ very general hypersurface.

Theorem (Kollár; 1995)

If $d \geq 2 \cdot \lceil (n+3)/2 \rceil$ then X is not ruled (in particular not rational).

- This was generalized and improved by Totaro and subsequently by Schreieder.

Mori's degeneration

Let R be a DVR with residue field κ and fraction field η.

There is a family

$$\mathcal{X} \to T := \operatorname{Spec} R$$

of weighted complete intersections such that

$$\begin{aligned} X_\eta \subset \mathbb{P}^{n+1}_\eta : \quad & \text{hypersurface of degree } pe; \\ X_0 : \quad & \text{degree } p \text{ cyclic cover of a} \\ & \text{smooth degree } e \text{ hypersurface } Y \subset \mathbb{P}^{n+1}_\kappa. \end{aligned}$$

Special behavior: when R is a mixed char $p > 0$ DVR!

Kollár's argument

There is a resolution of the special fiber

$$\begin{array}{ccccc} X_0' & \xrightarrow{\mu} & X_0 & \longrightarrow & Y \subset \mathbb{P}^{n+1}_\kappa \end{array}, \qquad \nu : X_0' \to Y$$

Assuming $n \geq 3$, Kollár shows that

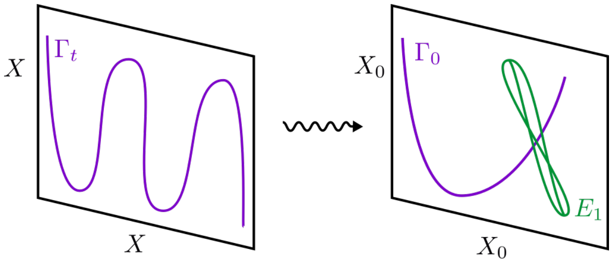
$$\underset{\text{big \& nef}}{\mathcal{M}} \hookrightarrow \wedge^{n-1}\Omega_{X_0'} \quad \text{and} \quad H^0(\wedge^{n-1}\Omega_{X_0'}) \neq 0.$$

- In char $p > 0$, these special Fano hypersurfaces $\mathcal{X}_0$ carry forms which have some additional positivity, hence X_0 not ruled.
- Ruledness specializes well, so when we lift to characteristic 0, we get hypersurfaces that are not ruled.

Sketch of main result

We will use Mori's construction.

Step 1. Start with $\phi_t \in \mathrm{RatEnd}(X_t) \iff \Gamma_t \subset X_t \times X_t$. Degenerate to positive characteristic:

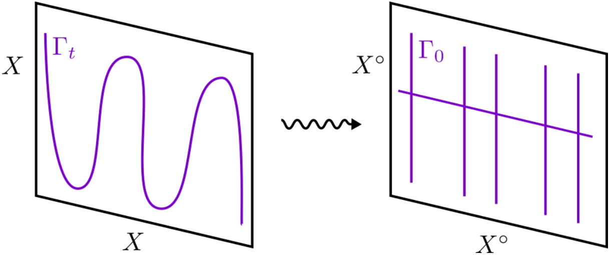


The graph on the central fiber possibly breaks up into a union of components

$$\Gamma_0 \cup E_1 \cup \cdots \cup E_n.$$

In practice, it is possible for graphs to break terribly:

Consider a family of maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree *d* which degenerates into a union of vertical fibers and one horizontal fibers.



This sort of degenerate behavior is special to $\mathbb{P}^1$ (and ruled varieties).

Singularities

X_R = normal scheme over R = DVR.

Definition. X_R has *sustained ruled modifications* if for every finite extension of DVRs $R \subset R'$, there is a further finite extension $R' \subset R''$ such that $X_{R''}$ is a normal scheme with ruled modifications.

Ex. (Abhyankar) A regular scheme X admits admits ruled modifications.[1]

This can be checked locally - suffices to find a proper birational map $\bar{X} \to X$ from a regular scheme...

Relevance:

- Given a family X_R, this will allow us to specialize properties from the *geometric* generic fiber $X_{\bar{\eta}}$ to the special fiber X_0.

[1] Ruled modifications: every exceptional divisor of every normal birational modification $Y \to X$ is ruled.

Recall: The graph on the central fiber possibly breaks up into a union of components

$$\Gamma_0 \cup E_1 \cup \cdots \cup E_n \subset X_0 \times X_0.$$

Theorem (Kollár)

X_0 is not separably uniruled.

Step 2. Technical part: give a partial resolution of cyclic covers in mixed characteristic. After allowing **arbitrary** base change, we can show:

Prop: All exceptional divisors E_i are separably uniruled.

$$\implies \deg\left(\pi_2\big|_{E_i} : E_i \dashrightarrow X_0\right) \equiv 0 \pmod{p}.$$

Note: We do not give a resolution of Mori's family. Let $R = \mathbb{Z}_p^{\mathrm{sh}}$.

Let $X_R \to Y_R$ be the p-cyclic cover branched over D. This like the central fiber in Mori's family but in mixed characteristic.

Theorem. X_R has sustained separably uniruled modifications.

- The proof involves a few types of weighted blow-ups. We find a birational model of X_R which has cyclic quotient singularities by groups of small order.

$$X_d \subset \mathbb{P}^{n+1}_{\mathbb{C}} \underset{\text{extra step}}{\rightsquigarrow} X_0 \to Y_0 \subset \mathbb{P}^{n+1}_{\mathbb{C}} \rightsquigarrow X_{\overline{\mathbb{F}}_p} \to Y_{\overline{\mathbb{F}}_p} \subset \mathbb{P}^{n+1}_{\overline{\mathbb{F}}_p}.$$

Step 3. Find an obstruction to separable rational endomorphisms on central fiber.

Let Z be a $\mathbb{Q}$-factorial variety.

Proposition. Suppose $M \hookrightarrow \wedge^i \Omega_Z$ $(i > 0)$ is a big and nef line bundle. Then any separable rational endomorphism $\phi\colon Z \dashrightarrow Z$ has degree 0 or 1.

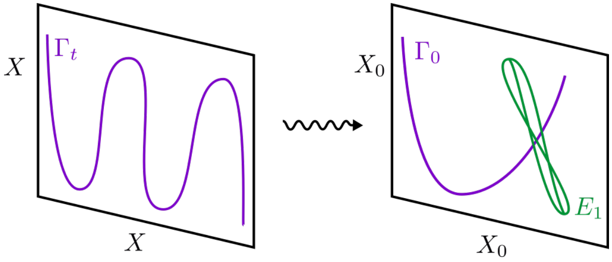
Consequence:

- X_0 does not admit separable rational endomorphisms of degree ≥ 2.

Idea: recall

$$\underbrace{L}_{\text{big + nef}} \hookrightarrow \wedge^{n-1}\Omega_{X_0}.$$

Iterate $\phi \in \mathrm{RatEnd}(X)$ and pull back L under $\phi^{\circ k}$ to produce an arbitrarily positive line bundle inside $\wedge^{n-1}\Omega_{X_0}$, contradiction.



Theorem (C-Stapleton)

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If $\phi \in \mathrm{RatEnd}(X)$ has degree λ, then $\lambda \equiv 0$ or $1 \pmod{p}$.

Related work

Observe: elliptic fibrations (with a section) always have multiplication by n maps. This gives a rational endomorphism of degree n^2.

Fact: if $n^2 \equiv 0$ or $1 \pmod{p}$ for all $n \geq 2 \implies p = 2, 3$.

Corollary

$X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ very general. If $d \geq 5\lceil (n+3)/6 \rceil$, then $X \not\simeq_{bir.}$ to an elliptic fibration.

This is similar in spirit to:

Theorem (Kollár). $X_d \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ very general. If $d \geq 3\lceil (n+3)/4 \rceil$, then $X \not\simeq_{\mathrm{bir.}}$ conic bundle.

- For CY hypersurfaces, Grassi and Wen proved an even stronger statement: they do not admit genus 1 fibrations.

Thank you for listening!